Reg. No. | | | | | | | | |

MTS 455

II Semester M.Sc. Degree Examination, September/October 2022
(CBCS – New Syllabus)
MATHEMATICS
Linear Algebra – II

Time : 3 Hours Max. Marks : 70

Note : *1) Answer **any five full** questions.*
*2) Answer to **each full** question shall **not** exceed **eight** pages of the answer book. **No** additional sheets will be provided for answering.*
*3) **Use** of scientific calculator is **permitted**.*

1. a) Define a bilinear form. For $x = (x_1, x_2)$, $y = (y_1, y_2)$ in the vector space $\mathbb{R}^2$, define $B(x, y) = x_1y_2 - x_2y_1$. Is B a bilinear form ? Justify.

   b) Define the matrix of a bilinear form. If A and A′ are two matrices of a bilinear form with respect to two bases B and B′, then derive the relation between A and A′ in terms of the matrix of change of basis.

   c) Prove that the bilinear form represented by the matrix $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ is positive definite if and only if $a > 0$ and $ad - b^2 > 0$. **(4+6+4)**

2. a) Prove that the following statements are equivalent for an $n \times n$ real matrix :

   i) A is symmetric and positive definite.

   ii) The form X^tAY represents dot product, w.r.t. some basis of $\mathbb{R}^n$.

   iii) $A = P^t P$ for some $P \in GL_n(\mathbb{R})$.

   b) Prove that the eigen values of a Hermitian matrix are real numbers.

   c) If a Hermitian form on V is not identically zero, then show that there is a vector $v \in V$ such that $\langle v, v \rangle \neq 0$. **(9+2+3)**

P.T.O.

3. a) Prove the existence of an orthonormal basis in a finite dimensional vector space V with a positive definite Hermitian form.

   b) Let $\langle \, , \rangle$ be a symmetric form on a real vector space V and let W be a subspace of V on which $\langle \, , \rangle$ is non-degenerate. If $(w_1, w_2, ..., w_n)$ is a basis for W, then obtain the projection formula for V onto W.

   c) Let $W = \text{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ in $\mathbb{R}^3$. Determine the orthogonal projection of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ onto W. **(5+5+4)**

4. a) Define a Unitary operator on a Hermitian space V. Show that $T : V \to V$ is unitary if and only if $\langle Tv, Tw \rangle = \langle v, w \rangle \; \forall \; v, w \in V$.

   b) State and prove Spectral theorem for normal operators. **(4+10)**

5. a) Let $f : V \to V'$ be a surjective homomorphism of R-modules, with kernel W. Show that there is a one to one correspondence between submodules of V′ and submodules of V containing W.

   b) Define a finitely generated module and a free module. Are all finitely generated modules free ? Justify. **(9+5)**

6. a) Let R be a non-zero ring. Prove that any two bases of the same free module over R have the same cardinality.

   b) Show that any subgroup of a free abelian group is free. **(7+7)**

7. a) Let V be an R-module. Prove that every submodule of V is finitely generated if and only if submodules of V satisfies ascending condition in the inclusion order.

   b) Let $\phi : V \to V'$ be a homomorphism of R-modules. If the kernel and image of ϕ are finitely generated, then show that V is finitely generated. **(7+7)**

8. a) State and prove Hilbert basis theorem.

   b) Find matrix P such that $P^{-1}AP$ is a diagonal matrix, where

$$A = \begin{bmatrix} t^2 - 3t + 1 & t - 2 \\ (t-1)^3 & t^2 - 3t + 2 \end{bmatrix}.$$

(10+4)